

- 8.1** (a) Let \mathcal{D} be the domain in the Einstein cylinder $(\mathbb{R} \times \mathbb{S}^3, g_E)$, $g_E = -dt^2 + g_{\mathbb{S}^3}$, which is the image of Minkowski spacetime (\mathbb{R}^{3+1}, η) under the conformal map $(u, v) \rightarrow (\text{Arctan}(u), \text{Arctan}(v))$ that we saw in class. Let p be a point on future null infinity $\mathcal{I}^+ \subset \partial\mathcal{D}$ and consider the set $C^-(p) \cap \mathcal{D}$ of past null geodesics emanating from p restricted to \mathcal{D} . How does this set look like in the standard Cartesian coordinates of \mathbb{R}^{3+1} ? Deduce that every pair of null geodesics of (\mathbb{R}^{3+1}, η) asymptoting in the future to the same point on \mathcal{I}^+ have to asymptote in the past to the same point on \mathcal{I}^- .
- (b) Let (\mathcal{M}^{3+1}, g) be a spherically symmetric spacetime without axis, i.e. $\mathcal{M}^{3+1} = \mathcal{U}^{1+1} \times \mathbb{S}^2$ and, in any local coordinates (x^1, x^2) on \mathcal{U} ,

$$g = \tilde{g}_{AB} dx^A dx^B + r^2 g_{\mathbb{S}^2},$$

where \tilde{g} is a Lorentzian metric on \mathcal{U} and $r : \mathcal{U} \rightarrow (0, +\infty)$ is a smooth function. The spacetime (\mathcal{U}, \tilde{g}) is known as the *Penrose diagram* of (\mathcal{M}, g) and can be formally thought of as the projection $\mathcal{U} = \mathcal{M}/SO(3)$. Show that the image $\tilde{\gamma}$ in \mathcal{U} of a causal curve γ in \mathcal{M} is again a causal curve (hence, Penrose diagrams are useful 2-dimensional tools to read-off the causal structure of a 4 dimensional spacetime). In which case is the projection of a null curve in \mathcal{M} again a null curve in \mathcal{U} ?

- 8.2** (a) Show that the Schwarzschild metric

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

on $\mathcal{M} = \mathbb{R}_t \times (2M, +\infty)_r \times \mathbb{S}^2$ is indeed a solution of the vacuum Einstein equations.

- (b) Show that (\mathcal{M}, g_M) embeds isometrically into $\tilde{\mathcal{M}} = \mathbb{R}_{t^*} \times (0, +\infty)_r \times \mathbb{S}^2$ with

$$g_{\tilde{\mathcal{M}}} = -\left(1 - \frac{2M}{r}\right) (dt^*)^2 + \frac{4M}{r} dt^* dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

(*Hint: Use the coordinate transformation $t^* = t + f(r)$ for a suitable $f(r)$.) Show that, in the extended spacetime, the region $\{r \leq 2M\}$ corresponds to a *black hole*, that is to say, no future directed causal curve starting from $\{r \leq 2M\}$ can end up in the asymptotically region $\{r \gg 1\}$.*

- 8.3** (a) Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a geodesic in the Schwarzschild (exterior) spacetime (\mathcal{M}, g_M) . Show that the geodesic equation takes the form

$$\begin{aligned} \frac{d}{ds} \left(\left(1 - \frac{2M}{r}\right) \dot{t} \right) &= 0, \\ \frac{d}{ds} \left(\left(1 - \frac{2M}{r}\right)^{-1} \dot{r} \right) &= \frac{1}{2} \left(-\frac{2M}{r^2} \dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-2} \frac{2M}{r^2} \dot{r}^2 + 2r\dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2 \right), \\ \frac{d}{ds} (r^2 \dot{\theta}) &= \frac{1}{2} r^2 \sin \theta \cos \theta \dot{\phi}^2, \end{aligned}$$

$$\frac{d}{ds} \left(r^2 \sin^2 \theta \dot{\phi} \right) = 0.$$

Deduce that one can without loss of generality one can consider geodesics lying in the equatorial plane $\theta = \frac{\pi}{2}$ (by possibly rotating the coordinate system (θ, ϕ) on S^2 .) Note that, in this case, the first and fourth of the equations above reduce to the statement that the energy E and angular momentum L of a geodesic are constant (i.e. are constants of motion for the geodesic flow).

- (b) Show that there exist “trapped” null geodesics orbiting the black hole (i.e. null geodesics that never approach $r = 2M$ or $r = \infty$) (*Hint: For an appropriately chosen value of $r = r_0 > 2M$, show that there exist null geodesics with $r(s) = r_0$ for all s .* Contrast this with the situation on Minkowski spacetime.

Remark. The region traced out by trapped null geodesics consists the so-called *photon sphere* of a black hole.

- (c) Show that, for any $\mu > 0$, there exist timelike geodesics γ in the Schwarzschild spacetime with $g(\dot{\gamma}, \dot{\gamma}) = -\mu$ which are trapped. (*Hint: It might be convenient, instead of working with the second order equations, to use the invariants of the geodesic flow and obtain a relation for \dot{r} and observe that, for an appropriate choice of E, L and $r(0)$, $r(s)$ cannot escape a bounded interval in r .)*

Remark. These timelike orbits correspond to massive objects (e.g. planets) moving under the influence of gravity in Schwarzschild spacetime. Unlike the trapped null geodesics, these orbits are *stable*, namely they remain trapped even under small perturbations of the initial condition $r(0)$ and the conserved quantities E, L, μ (were these trapped orbits not stable, earth would plunge in the sun under small perturbations of its orbit).